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A SIMPLE PROOF OF SOME ERGODIC THEOREMS

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ABSTRACT

Some ideas of T. Kamae's proof using nonstandard analysis are employed to give a simple proof of Birkhoff's theorem in a classical setting as well as Kingman's subadditive ergodic theorem.

Let $(X,\mathcal{B},\mu)$ be a probability measure space and let $T:X\to X$ be a measurable, measure preserving transformation, possibly noninvertible. Birkhoff's ergodic theorem states that for any integrable function f, the limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}f(T^jx)=f^*(x)$$

exists for μ-a.e. x, and f^* is a T-invariant function with the same integral as f. We adapt an idea of T. Kamae [1] to give a simple proof of this result. It is sufficient to deal with nonnegative functions and defining

$$\bar{f}(x)=\limsup_{n\to\infty}\frac{1}{n}\sum_{0}^{n-1}f(T^jx),\quad \underline{f}(x)=\liminf_{n\to\infty}\frac{1}{n}\sum_{0}^{n-1}f(T^jx)$$

it suffices to show that

$$\int\bar{f}(x)d\mu(x)\leqq\int f(x)d\mu(x)\leqq\int\underline{f}(x)d\mu(x)$$

since that gives equality a.e. $\bar{f}(x)=\underline{f}(x)$ and $\int f^*d\mu=\int fd\mu$ while the T-invariance of both $\bar{f}$ and $\underline{f}$ is clear. Fix some $M>0$, $\varepsilon>0$, denote

$$\bar{f}_M(x)=\min\{\bar{f}(x),M\}$$

and define $n(x)$ to be the least integer $n\geqq 1$ for which

$$\bar{f}_M(x)\leqq\frac{1}{n}\sum_{0}^{n-1}f(T^jx)+\varepsilon.$$

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Since $\bar{f}$ is T-invariant so is $\bar{f}_M$ and thus averaging gives that for all x

$$\sum_0^{n(x)-1} \bar{f}_M(T^jx) \leqq \sum_0^{n(x)-1} f(T^jx) + n(x)\cdot\varepsilon. \tag{1}$$

Now $n(x)$ is everywhere finite so that there is some N for which the set

$$A = \{x : n(x) > N\}$$

has measure less than ε/M. Define now

$$\tilde{f}(x) = \begin{cases} f(x), & x \notin A, \\ \max\{f(x), M\}, & x \in A, \end{cases} \qquad \tilde{n}(x) = \begin{cases} n(x), & x \notin A, \\ 1, & x \in A, \end{cases}$$

and observe that

$$\sum_0^{\tilde{n}(x)-1} \bar{f}_M(T^jx) \leqq \sum_0^{\tilde{n}(x)-1} \tilde{f}(T^jx) + \tilde{n}(x)\cdot\varepsilon \tag{$\tilde{1}$}$$

is also valid. The crucial improvement is that now $\tilde{n}(x)$ is everywhere bounded by N, while

$$\int \tilde{f}(x)d\mu(x) \leqq \int f(x)d\mu(x) + \int_A M\cdot d\mu(x) \leqq \int f(x)d\mu(x) + \varepsilon. \tag{2}$$

Choosing now L so that $NM/L < \varepsilon$ and defining inductively $n_0(x) = 0$ and

$$n_k(x) = n_{k-1}(x) + \tilde{n}(T^{n_{k-1}(x)}x), \cdots$$

we have

$$\sum_0^{L-1} \bar{f}_M(T^jx) = \sum_{k=1}^{k(x)} \sum_{n_{k-1}(x)}^{n_k(x)-1} \bar{f}_M(T^jx) + \sum_{n_{k(x)}(x)}^{L-1} \bar{f}_M(T^jx)$$

where $k(x)$ is the maximal k for which $n_k(x) \leqq L-1$. Applying ($\tilde{1}$) to each of the $k(x)$ terms, and estimating by M the last $L - n_{k(x)}(x) \leqq N-1$ terms we have for all x

$$\sum_0^{L-1} \bar{f}_M(T^jx) \leqq \sum_0^{L-1} \tilde{f}(T^jx) + L\cdot\varepsilon + (N-1)M,$$

where the fact that $\tilde{f} \geqq 0$ allows us to write $L-1$ as the upper limit of the summation. Integrating both sides and dividing by L gives

$$\int \bar{f}_M d\mu \leqq \int \tilde{f} d\mu + \varepsilon + \frac{(N-1)M}{L} \leqq \int f d\mu + 3\varepsilon$$

in light of (2) and the choice of L. It is here that we use the fact that T is measure preserving. Letting $\varepsilon \to 0$ and $M \to \infty$ gives half of what we wanted, namely

$$\int \bar{f} d\mu \leqq \int f d\mu.$$

For the other half, fix $\varepsilon > 0$ and define now $n(x)$ as the least integer $n \geqq 1$ for which

$$\frac{1}{n} \sum_{0}^{n-1} f(T^j x) \leqq \underline{f}(x) + \varepsilon.$$

As before $A = \{x : n(x) > N\}$ where now N is chosen so that $\int_A f(x) d\mu(x) < \varepsilon$. We define now

$$\tilde{n}(x) = \begin{cases} n(x), & x \notin A, \\ 1, & x \in A, \end{cases} \qquad \tilde{f}(x) = \begin{cases} f(x), & x \notin A, \\ 0, & x \in A, \end{cases}$$

and conclude the proof in the same way as before.

Observe that we could have restricted the integration to any T-invariant set so that we really have shown that $f^*(x)$ is a version of the conditional expectation of f with respect to the σ-algebra of invariant sets. The same basic idea, of modifying the function so that $n(x)$ becomes bounded, can be used to simplify proofs of other ergodic theorems as well. To illustrate the possibilities we give such a proof of Kingman's subadditive ergodic theorem [2]:

THEOREM. *If T is a measure preserving transformation of the probability measure space $(X, \mathcal{B}, \mu)$ and $\{f_n\}_1^\infty$ is a sequence of L^1-functions satisfying*

$$f_{n+m}(x) \leqq f_n(x) + f_m(T^n x), \qquad \text{all } n, m \geqq 1 \tag{3}$$

then $\lim_{n\to\infty} (1/n) f_n(x)$ *exists* a.e. *and may be identified as* $\phi(x) = \inf_n (1/n) f_n^*(x)$ *where f_n^* is the projection of f_n onto the space of T-invariant functions.*

For the proof, note first that (3) implies

$$f_{n+m}^*(x) \leqq f_n^*(x) + f_m^*(x) \tag{3'}$$

and hence $(1/n) f_n^*(x)$ converges to $\phi(x)$. Next, denote

$$\bar{f}(x) = \limsup \frac{1}{n} f_n(x), \qquad \underline{f}(x) = \liminf \frac{1}{n} f_n(x),$$

and observe that both $\bar{f}$ and $\underline{f}$ are T-invariant. Now

$$\frac{1}{n} f_n(x) \leqq \frac{1}{n} \sum_{j=0}^{n-1} f_1(T^j x) \tag{4}$$

and thus by Birkhoff's ergodic theorem $\bar{f}(x) \leqq f_1^*(x)$.

We remark at this point that (4) implies that the sequence $\{(1/n) f_n^+\}$ is equi-integrable, and combining this with the obvious inequality

$$\int \phi d\mu \leqq \int \frac{1}{n} f_n^* d\mu = \int \frac{1}{n} f_n d\mu, \qquad \text{all } n,$$

we see that if $\int \phi d\mu > -\infty$, then the pointwise convergence a.e. of $(1/n) f_n$ to ϕ implies convergence in L^1-norm. We have a similar, asymptotic, estimate with f_N instead of f_1 in (4). Fix $N > 1$ and let $n > N$. For each $i = 1, 2, \cdots, N$ write $n = i + mN + k$ with $k < N$. Then by (3)

$$f_n(x) \leqq f_i(x) + \sum_{l=0}^{m-1} f_N(T^{lN+i} x) + f_k(T^{mN+i} x)$$

and summing over i,

$$N f_n(x) \leqq \sum_{i=1}^{N-1} f_i(x) + \sum_{j=0}^{n-1} f_N(T^j x) + \sum_{i=1}^{N} f_{n-i-mN}(T^{mN+i} x)$$

hence

$$\frac{1}{n} f_n(x) \leqq \frac{1}{nN} \sum_{j=0}^{n-1} f_N(T^j x) + \frac{1}{nN} \left(\sum_{i=1}^{N} f_i(x) + \sum_{i=1}^{N} f_{n-i-mN}(T^{mN+i} x) \right).$$

As $n \to \infty$ the last two terms on the right converge to zero a.e. and, by the ergodic theorem,

$$\bar{f}(x) \leqq \frac{1}{N} f_N^*(x) \quad \text{a.e.}$$

which implies

$$\bar{f}(x) \leqq \phi(x) \quad \text{a.e.} \tag{5}$$

For points x where $\phi(x) = -\infty$, (5) shows that the desired limit exists and equals $\phi(x)$. We restrict our attention to $X_M = \{x : \phi(x) \geqq -M\}$, which is T-invariant, and proceed to show that

$$\int_{X_M} \underline{f} d\mu \geqq \int_{X_M} \phi d\mu. \tag{6}$$

This combined with (5) shows that the statement of the theorem is valid on X_M, and as $\bigcup_1^\infty X_M = \{x : \phi(x) > -\infty\}$ this will complete the proof. For ease of notation we will simply assume $\phi(x) \geqq -M$ for all x.

As in the proof of the Birkhoff theorem fix an $\varepsilon > 0$, set $\underline{f}_M = \max\{\underline{f}, -M-1\}$, and put

$$n(x) = \min\left\{n \geqq 1 : \frac{1}{n} f_n(x) \leqq \underline{f}_M(x) + \varepsilon\right\}.$$

Set $A = \{x : n(x) > N\}$ where N is chosen so that

$$\int_A (|f_1(x)| + M + 1)d\mu(x) < \varepsilon, \tag{7}$$

and define the modifications as before:

$$\tilde{\underline{f}}_M(x) = \begin{cases} \underline{f}_M(x), & x \notin A, \\ f_1(x), & x \in A, \end{cases} \qquad \tilde{n}(x) = \begin{cases} n(x), & x \notin A, \\ 1, & x \in A. \end{cases}$$

Note that $\tilde{\underline{f}}_M(x) \leqq \underline{f}_M(x)$ for all x, and by (7)

$$\int \tilde{\underline{f}}_M d\mu \leqq \int \underline{f}_M d\mu + \varepsilon. \tag{8}$$

Using the T-invariance of $\underline{f}_M$ we have for all x

$$f_{\tilde{n}(x)}(x) \leqq \sum_{j=0}^{\tilde{n}(x)-1} \tilde{\underline{f}}_M(T^j x) + \tilde{n}(x) \cdot \varepsilon$$

and can calculate for any $L > N$ as before:

$$f_L(x) \leqq \sum_0^{L-1} \tilde{\underline{f}}_M(T^j x) + L \cdot \varepsilon + N(M+1) + \sum_{L-N}^{L-1} |f_1(T^j x)|.$$

Integrating and dividing by L we obtain

$$\int \phi(x) d\mu \leqq \int \frac{1}{L} f_L^* d\mu = \int \frac{1}{L} f_L d\mu$$
$$\leqq \int \tilde{\underline{f}}_M d\mu + \varepsilon + \frac{N(M+1)}{L} + \frac{N}{L} \cdot \int |f_1| d\mu.$$

Letting $L \to \infty$ and using (8), we see that

$$\int \phi(x) d\mu \leqq \int \underline{f}_M(x) d\mu. \tag{9}$$

Recall now that $\underline{f}_M(x) \leqq \phi(x)$ holds for all x and that, combined with (9), implies

$\underline{f}_M(x) = \phi(x)$ a.e. Since whenever $\underline{f}_M(x) \neq \underline{f}(x)$ we have $\underline{f}_M(x) = -M - 1 \neq \phi(x)$, this can happen only on a null set and $\underline{f}(x) = \phi(x)$ a.e.

References

1. T. Kamae, *A simple proof of the ergodic theorem using nonstandard analysis*, Isr. J. Math. **42** (1982), 284–290.

2. J. F. C. Kingman, *Subadditive ergodic theory*, Ann. Probab. **1** (1973), 883–909.

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